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# Poisson's remarkable calculation – a method or a trick?

Denis Bell

Denis Bell was awarded his doctorate degree from the University of Warwick. He is currently employed as a Professor of Mathematics at the University of North Florida. His area of research is stochastic analysis and the application of probabilistic methods to problems in classical analysis.

The Gaussian function $e^{-x^2}$ plays a fundamental role in probability and statistics. For this reason, it is important to know the value of the integral

$$I = \int_0^\infty e^{-x^2} dx.$$

Since the integrand does not have an elementary antiderivative, $I$ cannot be evaluated directly by the fundamental theorem of calculus. The familiar computation of the Gaussian integral is via the following remarkable trick, attributed to Poisson. One forms the square of $I$, interprets it as a double integral in the plane, transforms to polar coordinates and the answer magically pops out. The calculation is as follows

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^{\pi/2} \int_0^\infty r e^{-r^2} dr d\theta = \frac{\pi}{2} \int_0^\infty r e^{-r^2} dr = \pi/4.$$

Hence $I = \sqrt{\pi}/2$.

Den meisten Leserinnen und Lesern unserer Zeitschrift dürfte die vermutlich auf Poisson zurückgehende Methode zur Berechnung des Integrals $I$ über die Gaußsche Fehlerfunktion $\exp(-x^2)$ entlang der reellen Zahlengeraden bekannt sein: Man bildet dazu das Flächenintegral $I^2$ und führt Polarkoordinaten ein; das fragliche Integral lässt sich nun elementar berechnen. In dem vorliegenden Beitrag ermittelt der Autor alle auf $\mathbb{R}$ stetigen und integrierbaren Funktionen $f$, für welche das Integral $\int_{\mathbb{R}} f(x)\,dx$ mit dieser Methode berechnet werden kann. Es stellt sich heraus, dass dies bis auf Skalierung die Funktionen der Form $f(x) = x^p \exp(cx^2)$ mit $p > -1$ und $c < 0$ sind.

In this article we explore the question: *can Poisson's method be used to compute other seemingly intractable integrals?* To this end, consider the improper integral

$$J = \int_0^\infty f(x)dx.$$

If $f$ satisfies the functional equation

$$f(x)f(y) = g(x^2 + y^2)h(y/x), \quad x, y > 0 \tag{1}$$

then Poisson's argument applied to the function $f$ yields

$$J^2 = \frac{1}{2}\Big(\int_0^\infty g(x)dx\Big)\Big(\int_0^{\pi/2} h(\tan\theta)d\theta\Big).$$

The determination of $J$ is thereby reduced to the evaluation of two, hopefully more elementary and solvable, integrals. This procedure raises the question:

*Which functions $f$ satisfy an equation of the form* (1) *where the functions $g$ and $h \circ \tan$ admit elementary antiderivatives?*

This problem was studied by the author in [3] where it was shown that the only continuous solutions $f$ to (1), assumed to be asymptotic to $x^p$ at zero, are those of the form $f(x) = ax^pe^{cx^2}$. R. Dawson [4] proved a similar result for the less general equation

$$f(x)f(y) = g(x^2 + y^2), \quad x, y \geq 0,$$

assuming only Riemann integrability of $f$. In the present article, we prove the following generalization of these results.

**Theorem**. *Suppose $f : (0, \infty) \mapsto \mathbb{R}$ satisfies equation* (1)*, $f$ is non-zero on a set of positive Lebesgue measure, and the discontinuity set of $f$ is not dense in* $(0, \infty)$*. Then $f$ has the form*

$$f(x) = Ax^pe^{cx^2} \tag{2}$$

*where $A$, $p$ and $c$ are constants. Furthermore, the functions $g$ and $h$ are unique up to scalar multiplication and are given by*

$$g(x) = A_1x^pe^{cx}, \tag{3}$$

$$h(x) = A_2\Big(\frac{x}{1+x^2}\Big)^p \tag{4}$$

*where $A_1A_2 = A^2$.*

Hence the only reasonable functions one might hope to integrate by Poisson's method are those of the form (2). Denoting

$$J = \int_0^\infty x^pe^{cx^2}dx,$$

the decomposition into (3) and (4) implied by the theorem results in the expression

$$J^2 = \frac{1}{2^{p+1}} \int_0^\infty x^p e^{cx}\, dx \;\times\; \int_0^{\pi/2} \sin^p t\, dt.$$

Now the existence of the first integral requires $p > -1$ and $c < 0$, while the evaluation of both integrals in closed form requires that $p$ be an *integer*. But in this case, the computation of $J$ can be evaluated in terms of the Gaussian integral $I$ by elementary techniques (i.e., substitution and integration by parts)! We conclude that *Poisson's argument has no wider applicability as an integration method*. This answers the question posed in the title of the article. As Dawson observes, it is curious that Poisson's remarkable calculation turns out to have essentially only one application and that this single application is such a significant one.

The proof of the theorem differs substantially from the argument in [3] in focussing on the function $g$ in (1) rather than on $h$. The proof will require three preliminary results.

**Lemma 1**. *Suppose $f$ satisfies* (1) *and $f$ is non-zero on a set of positive Lebesgue measure. Then $f$ never vanishes.*

*Proof*. Note first that $h(1) \neq 0$ otherwise taking $y = x$ in (1) gives $f \equiv 0$, contradicting the hypothesis. We suppose throughout, without loss of generality, that $h(1) = 1$. Setting $y = x$ in (1) gives

$$f^2(x) = g(2x^2). \tag{5}$$

Substituting for the function $g$ in (1), we obtain

$$f(x)f(y) = f^2\Big(\sqrt{\frac{x^2+y^2}{2}}\Big)h\Big(\frac{y}{x}\Big). \tag{6}$$

Define

$$r(x) = f(\sqrt{x}), \quad k(x) = h(\sqrt{x}).$$

Then (6) yields

$$r(x)r(tx) = r^2\Big(\frac{x(1+t)}{2}\Big)k(t), \quad t, x > 0. \tag{7}$$

We now prove the *claim: There exists $\delta > 0$ such that $k(x) \neq 0$ for all $x$ in the interval* $(1-\delta, 1+\delta)$. We argue by contradiction. Since the map $x \mapsto x^2$ is strictly monotone on $(0, \infty)$, the non-zero set of $r$ has positive Lebesgue measure $\lambda$. Hence there exists an integer $N$ such that $a := \lambda(T) > 0$ where $T$ denotes the set

$$\{x \mid r(x) \neq 0\} \cap [N, N+1].$$

*Suppose the claim does not hold*. Then there exists a sequence $t_n \to 1$ such that $k(t_n) = 0$ for all $n$. Equation (7) yields

$$r(x)r(t_n x) = 0, \quad \forall n,\ \forall x. \tag{8}$$

Define

$$T_n := \{t_n x \mid x \in T\}$$

and

$$V := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} T_k.$$

Then (8) implies $T_n$ and $T$ are disjoint, for all $n$. Hence

$$V \cap T = \emptyset. \tag{9}$$

Furthermore

$$\lambda(V) = \lim_n \lambda\Big(\bigcup_{k=n}^{\infty} T_k\Big) \geq \lim_n \lambda(T_n) = \lim_n t_n \lambda(T) = a. \tag{10}$$

If $x \in V$, then there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and $\{x_k\} \subset T$ such that $x = t_{n_k} x_k$ for all $k$. Since $t_{n_k} \to 1$, this implies $x \in \bar{T}$, i.e. $V \subset \bar{T}$.

Let $U$ be an arbitrary open set such that $T \subset U$. Then

$$T \cup V \subset \bar{U}. \tag{11}$$

We conclude from (9)–(11) that

$$\lambda(U) = \lambda(\bar{U}) \geq \lambda(T \cup V) = \lambda(T) + \lambda(V) \geq 2a.$$

Thus

$$a = \lambda(T) = \inf\{\lambda(U) \mid U \text{ open},\ T \subset U\} \geq 2a.$$

This implies $a = 0$, a contradiction. The claim follows.

Now suppose $f$ vanishes, so $r(x_0) = 0$ for some $x_0 > 0$. Setting $x = x_0$ in (7) and using the claim, we deduce that $r \equiv 0$ on the interval $x_0(1 - \delta/2, 1 + \delta/2)$. Extrapolating this property results in the conclusion $r \equiv 0$, hence $f \equiv 0$ which contradicts the hypothesis of the lemma. □

Before proceeding further, we give some examples of functions $f$ satisfying equation (1) that *do not have the form* (2).

**Example 1**. Let $m$ be a *discontinuous* function on $(0, \infty)$ with the multiplicative property

$$m(x)m(y) = m(xy), \quad x, y > 0. \tag{12}$$

(The existence of a large class of such functions is well-established, see e.g. [1]). Then equation (1) is satisfied with $f = g = m$ and $h(t) = m(t/(1 + t^2))$. Note that that $m$ is *never zero* (otherwise (12) implies $m \equiv 0$).

**Example 2**. Let $a > 0$ and define $f = I_{\{a\}}$, where $I_{\{a\}}$ denotes the indicator function of the singleton set $\{a\}$, $g = I_{\{2a^2\}}$, and $h = I_{\{1\}}$. Then it is clear that these functions satisfy (1).

**Example 3**. Let $\mathbf{A}$ denote the set of *algebraic* numbers in $(0, \infty)$. Define $f = g = h = I_{\mathbf{A}}$. Then (1) follows from the fact that $\mathbf{A}$ is closed under the positivity-preserving arithmetic operations and the extraction of square roots. Indeed, this immediately implies that if $I_{\mathbf{A}}(x)I_{\mathbf{A}}(y) = 1$, then $I_{\mathbf{A}}(x^2 + y^2)I_{\mathbf{A}}(y/x) = 1$. Conversely, suppose $I_{\mathbf{A}}(x^2 + y^2)I_{\mathbf{A}}(y/x) = 1$. Write $x^2 + y^2 = \alpha$ and $y/x = \beta$ where $\alpha, \beta \in \mathbf{A}$. Solving for $x$ and $y$, we have

$$x = \sqrt{\frac{\alpha}{1+\beta^2}}, \quad y = \beta\sqrt{\frac{\alpha}{1+\beta^2}}.$$

Thus $x, y \in \mathbf{A}$ and so $I_{\mathbf{A}}(x)I_{\mathbf{A}}(y) = 1$.

This example in conjunction with Lemma 1, provides a new proof of the well-known fact that (assuming at least one transcendental number exists) *the set of algebraic numbers has zero Lebesgue measure*. In fact, replacing $\mathbf{A}$ in this argument by an arbitrary set, we obtain the following result.

**Proposition**. *Let $E$ be a measurable proper subset of* $(0, \infty)$ *closed under addition, multiplication, division, and the extraction of square roots. Then $E$ has zero Lebesgue measure.*

The above examples show that neither of the additional hypotheses in the theorem is redundant.

**Lemma 2**. *Suppose $f$ satsifes the hypotheses of the theorem. Then $f$ is continuous everywhere.*

*Proof*. By assumption, there exists an interval $(a, b)$ on which $r$ is continuous. In view of Lemma 1, we may write (7) in the form

$$r(x) = \frac{r^2\left(\frac{x(1+t)}{2}\right)k(t)}{r(tx)}, \quad t, x > 0. \tag{13}$$

Suppose $x$ lies in the interval $(a, b) + 3(b - a)/4$. Choose and fix $t$ such that both $tx$ and $x(1 + t)/2$ lie inside $(a, b)$. Then (13) shows that $r$ is continuous at $x$. Iterating this property, we see that $r$ is continuous on $(a, \infty)$. A similar argument shows that $r$ is continuous on $(0, b)$. □

**Remark**. Lemma 2 implies that $f$ has constant sign, which we may suppose without loss of generality, is positive. We deduce from (7) that $k$ is then strictly positive and everywhere continuous.

**Lemma 3**. *Suppose $f$ satisfies the hypotheses of the theorem. Then the function* $\log r$ *is integrable at* 0.

*Proof*. The argument is a *quantitative* version of the iterative step in the proofs of Lemmas 1 and 2. We make repeated use of (13), which we write as

$$r(tx) = \frac{r^2\left(\frac{x(1+t)}{2}\right)k(t)}{r(x)}, \quad t, x > 0. \tag{14}$$

First, choose $\delta < .125$ and $l, L, m, M$ such that $0 < l, m < 1$, $M, L > 1$ and

$$l < r(x) < L, \quad x \in [1,2],$$
$$m < k(t) < M, \quad t \in [\delta, 1].$$

Taking $x = 2$ and letting $t$ vary in the range $[\delta, 1]$ in (14), we have

$$\frac{l^2 m}{L} < r(x) < \frac{L^2 M}{l}, \quad x \in [2\delta, 1]. \tag{15}$$

Now setting $x = 4\delta$ in (14) and using (15) yields

$$\frac{l^5 m^3}{L^2 M} < r(x) < \frac{L^5 M^3}{l^2 m}, \quad x \in [4\delta^2, 2\delta]. \tag{16}$$

Setting $x = 8\delta^2$ and using (16) in (14) we have

$$\frac{l^{12} m^8}{L^7 M^5} < r(x) < \frac{L^{12} M^8}{l^7 m^5}, \quad x \in [8\delta^3, 4\delta^2].$$

Note that the powers of $l, m, L, M$ in these estimates are increasing (roughly) by a factor of 3 each time. Iterating this process, we see that there exist constants $D < 1$ and $E > 1$ such that

$$D^{4^n} < r(x) < E^{4^n}, \qquad \epsilon^n < x < \epsilon^{n-1} \tag{17}$$

where $\epsilon = 2\delta$. Let

$$q = 4^{\frac{1}{\log \epsilon}}$$

and note that $q > 1/e$ by choice of $\delta$. Substituting $t = \epsilon^n$ in (17) gives

$$|\log r(x)| < q^{\log t} \max(-\log D, \log E), \quad x \in [t, t^{\frac{n-1}{n}}].$$

Since

$$\int_0^1 q^{\log t} dt = \int_{-\infty}^0 (eq)^x dx < \infty$$

this implies

$$\int_0^1 |\log r(x)| dx < \infty$$

and we are done. □

*Proof of the theorem.* Define

$$G(x) = \log r(x) - \frac{1}{x} \int_0^x \log r(u) du = \log r(x) - \int_0^1 \log r(xu) du, \quad x > 0. \tag{18}$$

(Note that Lemma 3 implies that the integrals exist.)

Taking logarithms in (13), we have

$$\log r(x) + \log r(tx) - 2\log r\Big(\frac{x(1+t)}{2}\Big) = \log k(t), \quad t, x > 0.$$

Thus

$$\begin{aligned} &G(x) + G(tx) - 2G\Big(\frac{x(1+t)}{2}\Big) \\ &\quad = \log k(t) - \int_0^1 \Big(\log r(xu) + \log r(txu) - 2\log r\Big(\frac{x(1+t)u}{2}\Big)\Big)\, du \\ &\quad = \log k(t) - \int_0^1 \log k(t) du = 0. \end{aligned}$$

Setting $y = tx$ gives

$$G(x) + G(y) = 2G\Big(\frac{x+y}{2}\Big), \quad x, y > 0. \tag{19}$$

Equation (19) is a variant of the Cauchy functional equation. It is well-known (and easy to show) that the only continuous functions $G$ satisfying (19) are *linear* functions $x \mapsto ax + b$. We can therefore write

$$\log r(x) - \frac{1}{x}\int_0^x \log r(u) du = \frac{cx + p}{2}$$

for constants $c$ and $p$. Multiplying by $x$ and differentiating yields

$$\frac{xr'(x)}{r(x)} = cx + \frac{p}{2}.$$

Solving for $r$ gives

$$r(x) = Ax^{p/2}e^{cx}.$$

Hence

$$f(x) = Ax^p e^{cx^2}$$

as claimed. Substituting this expression into (5) and (6), we obtain the functions $g$ and $h$ and the proof is complete. □

**Concluding remarks**

1. The methods of this paper can be used to treat the more general functional equation

$$f(x)g(y) = F(x^2 + y^2)G(y/x), \quad x, y > 0.$$

Assuming the discontinuity sets of $f$ and $g$ are non-dense and $f$ and $g$ are non-zero on sets of positive Lebesgue measure, we can show that (up to multiplicative constants) the functions necessarily have the form

$$f(x) = x^{p_1} e^{cx^2}, \quad g(x) = x^{p_2} e^{cx^2},$$
$$F(x) = x^p e^{cx}, \quad G(x) = \frac{x^{p_2}}{(1+x^2)^p},$$

where $p_1 + p_2 = 2p$.

2. This subject of functional equations, which originated with Cauchy and Abel, has spawned an extensive body of advanced techniques (see, e.g. [1]). These techniques have been used to prove far more general results than those presented here (cf. [2], [5], and [6]). The advantage of the present approach is that it provides a complete analysis of equation (1) in the present context, by direct and elementary means.
3. The problem addressed in this article admits a more general formulation. Consider an *arbitrary* change of coordinates $(x, y) \mapsto (u, v)$, where each of $u$ and $v$ depend on both $x$ and $y$. Which functions $f$ satisfy a product-preserving relation $f(x)f(y)dxdy = g(u)h(v)dudv$? We conjecture that (up to scaling) *there will generally exist a two parameter family of functions with this property*.

## References


[1] Aczel, J.: *Lectures on Functional Equations and Their Applications*. Academic Press, New York 1966.

[2] Baker, J.A.: On the functional equation $f(x)g(y) = p(x+y)q(x/y)$. *Aequationes Math.* 14 (1976), 493–506.

[3] Bell, D.: On the limitations of a well-known integration technique. *Math. Mag.* 66 (1993), 243–244.

[4] Dawson, R.: On a “singular” integration technique of Poisson. *Amer. Math. Monthly* 112 (2005) 3, 270–272.

[5] Lajko, K.: Remark to a paper by J.A. Baker. *Aequationes Math.* 19 (1979), 227–231.

[6] Meszaros, F.: *Numbers, functions, equations* 08. De La Motte Castle, Noszvaj, Hungary, 2008.



Denis Bell
Department of Mathematics
University of North Florida
1 UNF Drive
Jacksonville, FL 32224, USA
e-mail: dbell@unf.edu


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